Dedekind as a Philosopher of Mathematics

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Abstract Besides his celebrated work in algebra and number theory, Richard Dedekind made various technical contributions to the foundations of mathematics. This already qualifies him as an important figure in the philosophy of mathematics. But there are also other reasons to consider him a philosopher of mathematics, indeed a subtle, important, and still relevant one. In the present article, this will be illustrated by highlighting the logicist and structuralist aspects of Dedekind's approach to mathematics, as put in mathematical and historical context.

1 Introduction

In this article, I want to treat Richard Dedekind as a philosopher of mathematics. Is it reasonable, or plausible at all, to do so? There are several immediate reasons for skepticism. Dedekind was clearly a mathematician professionally, in terms of his training, his employment, and his professional connections. There is no place in his writings where he self-identifies as a philosopher; in fact, occasionally he distances himself from philosophy or at least makes light of it. If one takes it to be the mark of a philosopher to be concerned with "big questions", such as the basis of morality, the nature of reality, or the limits of human knowledge, he clearly does not qualify either; similarly if one has in mind "grand system building", as associated with figures such as Kant, Fichte, or Hegel. And even if one expects from a philosopher of mathematics only to address "smaller issues" relevant for that discipline, but to do so in an explicit and detailed way, as Frege, Russell, or Poincaré illustrate, it is hard to make the case for Dedekind as a philosopher.

Nevertheless, I will attempt to show that one can see Dedekind as a philosopher of mathematics, indeed a subtle, important, and still relevant one. But this will involve thinking of philosophy in a somewhat different, more

circumscribed way. For one thing, the philosophical questions he addresses were not ones imposed on mathematics from outside, as tends to be the case in "system-building" philosophy, but arose from within mathematical practice. For another, his answers are often only implicit in his writings, including in his technical results and his way of systematizing mathematics. Beyond that, there are several brief, but philosophically pregnant remarks by him that, if taken seriously, reveal themselves to form the tip of a philosophical iceberg. This becomes more evident once one also factors in philosophical influences on and by him, as well as the ways in which philosophical and mathematical developments were intertwined at the time.

2 Dedekind's Relation to Philosophy: Initial Observations

Let us start with some explicit references to philosophy in Dedekind's writings. In a letter to his sister Mathilde from June 11, 1852, he writes:

> But how can I get drawn into philosophy so much? What would Fichte say if he heard that one individual is posited before the others! The basic principle of his philosophy is, in fact: "The I posits itself." Think how smart a man must have been who derived our whole world order from that axiom. When I return I will give you a philosophical performance that will make your hair stand up. Unfortunately, I don't know too much about it. This ought to be a problem for me; but I do not take it too seriously. I can be quite happy without philosophy as long as I get my cello strings *citissime* ([11], p. 157, my translation).

Clearly Dedekind does not view himself as a thinker in the mold of Fichte. In fact, in this passage he seems to distance himself from philosophy, with its attempt to "derive the whole world order" from a single principle (and for mathematicians skeptical of philosophical pretensions, this may resonate strongly). Then again, what we have here with a light remark in a letter to a family member.

Fichte is not an easily accessible philosopher and it is remarkable that he is mentioned by Dedekind at all. We can find another reference to him two decades later, in Dedekind's obituary for his friend Bernhard Riemann published in 1876. In that context, Fichte is mentioned as an important influence on Riemann, as is J. F. Herbart. Herbart was a philosophy professor at the University of Göttingen until his death in 1841, not long before Dedekind's time as a student there (1850–1852). Dedekind himself did take a philosophy course with another well known Göttingen professor: Hermann Lotze. This was in the Summer Semester of 1852, when Lotze lectured on "German Philosophy since Kant" (as we know from notes Dedekind took that are preserved in his *Nachlass*)—and the course covered Kant, Fichte, Schelling, Hegel, and Herbart. So there is more to Dedekind's passing references to Fichte and Herbart than one might think initially.

Dedekind attended Lotze's lectures faithfully and presumably with real interest. At the same time, he never associates himself explicitly with any philosopher or philosophical school in his writings. Indeed, further references to philosophers are rare, so that we don't know much about which texts he read. Two exceptions occur in the Preface to *Was sind und was sollen die Zahlen?* (1888, cf. [8]). After mentioning that he became aware of Frege's most philosophical book, *Die Grundlagen der Arithmetik* (1884, [16]), only after having written his own text, Dedekind points out a close relationship between their goals. He also mentions Bernard Bolzano's *Paradoxien des Unendlichen* (1851, [3]), again to indicate that there was no direct influence, only some parallels. Beyond that, we know that Georg Cantor, as part of his correspondence with Dedekind in the 1880s, sent him some philosophical texts to read, including excerpts from Leibniz ([11], pp. 256-257); and there may have been a few further cases like that.

What these initial observations indicate is that Dedekind was more cognizant of philosophy than he may have let on. We should also remember that he lived during a time when some basic knowledge of philosophy was part of a general education, especially in Germany. However, if we focus only on direct ties to philosophers we miss the most important philosophical influence on him, which came from his teachers in mathematics. C.F. Gauss, Dedekind's dissertation advisor, was quite interested in philosophical questions, especially concerning geometry; similarly for Riemann, his colleague and friend. In fact, the whole Göttingen school in mathematics at the time—also including P.G.L. Dirichlet, Dedekind's other main mentor—approached mathematics in a philosophical manner. This was noted by the Göttingen physicist Wilhelm Weber, who pointed to "the fertile ground for the deep philosophical orientation in mathematical research that Göttingen became through Gauss, and has remained under Dirichlet and Riemann" ([11], p. 166, my translation). It remains to be clarified, however, how Dedekind fits into that research tradition and to which philosophical themes he was thereby led.

3 Foundational Results and Philosophical Themes

There is no question that Dedekind had a lasting influence on the philosophy of mathematics already in terms of several technical results, especially in *Was sind und was sollen die Zahlen?*, also in his earlier *Stetigkeit und irrationale Zahlen* (1872, [7]). To remind the reader briefly: In the latter, Dedekind offers a precise and elegant analysis of the notion of continuity (or line-completeness), clearly distinguishing it from denseness, in terms of his novel notion of a cut ("Dedekind cut"). Assuming the field of rational numbers, he then introduces the real numbers systematically via such cuts. In the former text, he provides an explicit definition of the infinity of sets ("Dedekind infinite"), and on that basis, a definition of the notion of a "simple infinity". He thereby characterizes the system of natural numbers via the Peano Axioms, or better, "Dedekind-Peano axioms" (although Dedekind does not talk about "axioms" and his characterization, in terms of the notion of "chain", is interestingly different from Peano's). He also proves that any two simple infinities are isomorphic (the axioms are categorical); and he provides a careful justification of definitions by recursion and proofs by induction.

These Dedekindian results have been central ingredients in almost all systematic accounts of the natural and real numbers since then. Not only is there a direct connection to Peano's work on the natural numbers, but also to Hilbert's and related axiomatizations of the real numbers (cf. [1]). Moreover, Ernst Zermelo built Dedekind's results into the core of axiomatic set theory, quite conscious of their origins (cf. [14]); and set theory became the dominant foundational framework for mathematics through the twentieth- and into the twenty-first century. But how did Dedekind see the philosophical relevance of these results himself, especially given his Göttingen background? Earlier I hinted at some brief but philosophically pregnant remarks in Dedekind's writings. These need to be taken into account now. Most of these remarks occur in his 1872 and 1888 booklets, although in his less foundational texts there are examples as well. In what follows, I will restrict myself to a few paradigmatic examples, which will lead us to two central philosophical themes in Dedekind's work.

At several places in the Preface to *Was sind und was sollen die Zahlen?* Dedekind states his goal of showing that arithmetic is "a part of logic", but he usually does so quite tersely. At one point he elaborates on it as follows:

> In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number concept entirely

> independent of the notion or *intuition of space and time*, that I consider it an immediate result from the *laws of thought* ([10], p. 31, my emphasis).

Several aspects of this passage are noteworthy. First, anyone familiar with Kant's philosophy will be struck by Dedekind's choice of words, especially the phrase "intuition of space and time". Kant had famously argued that all of mathematics, including arithmetic, is dependent on such intuition, and various later thinkers had picked up on that idea, approvingly or disapprovingly. Moreover, Dedekind surely noted this Kantian claim in the philosophy class he took with Lotze. Second, the parenthetical remark in the quoted passage shows that Dedekind means "arithmetic" in a broad sense, including "algebra" and "analysis". And indeed, at other points in this text he ties the discussion explicitly back to his 1872 essay on the real numbers. Hence, showing that arithmetic is "a part of logic" is a stronger goal than one might realize initially. Third, in a closely related passage, alluded to already earlier, Dedekind makes clear that he shares this goal with Frege. In other words, both saw themselves as "logicists" (without using that term). Notice, in addition, his use of the key phrase "laws of thought". Given all of that, Dedekind's logicism is a first philosophical theme deserving more attention.

Concerning a second such theme, let us turn to some striking remarks about "abstraction" and "free creation" by Dedekind. The most extended one occurs again in *Was sind und was sollen die Zahlen?*, now in the main body of the text:

> If in the consideration of a simply infinite system N set in order by a mapping ϕ we entirely neglect the special character of the elements, merely retaining their distinguishability and taking into account only their relations to one another, then are these elements called natural numbers [...]. With reference to this freeing the elements from every other content (*abstraction*) we are justified in calling numbers a *free creation of the human mind* ([10], p. 68, my emphasis, translation slightly amended).

At this point in the text, Dedekind has already defined the notions of simple infinity; he has also introduced a particular simple infinity N. His appeal to "abstraction" and "free creation" is then meant to make clear how he wants us to conceive of "the natural numbers" (thus answering the first half of the booklet's title question: "Was sind ... die Zahlen?"). And as he notes later on (Remark 134), this answer is further justified by his theorem that all simple infinities are isomorphic, so that the same arithmetic theorems hold for all of them.

How exactly to understand Dedekind's appeal to "abstraction" and "free creation" in this passage is controversial. In fact, it is tempting to simply ignore this appeal, among others because axiomatic set theory has taught us how to avoid it. It is worth adding, then, that Dedekind talks about "creation" at other places as well, including in *Stetigkeit und irrationale Zahlen*. The context there is the following: He has just introduced the system of all cuts on the rational numbers. He has also already shown that, endowed with suitable operations and relations, this system forms a complete order field. It is at that point—and again unlike in current set theory—that he tells us he does not mean to work with the cuts themselves. Instead, his procedure is to "create" new objects corresponding to them, which only deserve the name "the real numbers". Exactly this emphasis on the "creation" of novel objects is highlighted further in some of Dedekind's correspondence (cf. [22]). What we are dealing with here is his structuralist way of re-thinking the nature of mathematical objects, a second major philosophical theme.

4 Dedekind's Logicism: Background and Systematic Goal

The position of logicism is usually associated with the works of Frege and Russell. And indeed, I already mentioned that Dedekind points to a parallel between his project and Frege's. Logicism also tends to be seen as mainly a response to Kantian views about mathematics. Here too, I noted a connection. However, there is reason to think that in Dedekind's case the primary motivation has to do with the development of mathematics, or better, with debates concerning methodological and epistemological questions prompted by that development, rather than with "system building" philosophy (cf. [23], [25]).

The main influences on Dedekind in this connection are Gauss, Dirichlet, and Riemann. As further background, one needs to remember that until the beginning of the nineteenth century the foundations of mathematics, insofar as there were general ones, were assumed to reside in geometry. More specifically, arithmetic, algebra, and analysis were all taken to involve the notions of magnitude, and magnitudes were traditionally represented in terms of geometric constructions. Yet with the rise of various non-Euclidean geometries, Gauss was led to the question of whether arithmetic and geometry did not have very different foundations. He suggested that, while the correct geometry of "space" was ultimately an empirical matter, arithmetic should be seen as solely founded in more general "laws of thought"—the very term we saw Dedekind using. What I am pointing to here might be called "the birth of pure mathematics, as arithmetic" ([13]).

At the time, Gauss, W.R. Hamilton, and others had already shown how to treat complex numbers in terms of pairs of real numbers. Similarly, constructions of the negative and rational numbers via pairs starting with the natural numbers were familiar. This left three questions: (a) how to conceive of the real numbers, if not as based on a geometric notion of magnitude; (b) how to think of the natural numbers, assuming one did not just want to take them for granted; and (c) how to do both based on general "laws of thought" alone. Answering them was exactly the task Dedekind set for himself. Seen in this light, the goal of his logicism is a systematic re-thinking of all of "pure mathematics" in Gauss' sense.

There is a second mathematical development that motivated Dedekind's logicism as well. It consists of the rise of a kind of "conceptual" approach to mathematics that was promoted by Dirichlet and Riemann, in particular, partly also by Gauss, and was opposed to the more computational, formalistic mathematics characteristic of Weierstrass, Kronecker, and their Berlin school. The core goal here was to reconstruct various parts of mathematics by means of proofs from carefully defined, fundamental concepts, as opposed to relying on "blind calculations" (cf. [13], [25], earlier [31]). Dedekind was pursuing that goal too, especially as applied to the natural and real numbers; but he also contributed to the project more generally, including in his work in algebra and number theory.

Along such lines, one can think of Dedekind's logicism as a precursor to the "formal axiomatics" made prominent in Hilbert's subsequent work, besides influencing the use of axiomatic set theory as a foundation, along Zermelo's lines, very deeply. As such, his approach is opposed to "intuitionism", but not necessarily to "formalism". Indeed, this is a widespread view in the secondary literature on Dedekind (cf. [17], [28], [29], and partly [22]). But in addition, and perhaps more illuminatingly, one can see it as an original attempt to systematically re-think large parts of the new mathematics of the nineteenth century. In Dedekind's hands, that systematization took a logicist form, in the sense of identifying relevant concepts and laws fundamental to "thought" in general. In addition, his work always had a strongly dynamic character (cf. already [6]), insofar as he wanted to develop mathematics further, rather than providing a static foundation for it. In that respect, Dedekind's approach differs significantly from Frege's and Russell's.

5 Dedekind's Structuralism and Cassirer's Reception

The second philosophical theme identified above concerns Dedekind's structuralist conception of mathematical objects. This is perhaps an even more innovative contribution by him; but like his logicism, it is deeply rooted in methodological and systematic concerns. There is also again a direct connection to, and an influence on, Hilbert's later work. However, the distinctive character of Dedekind's structuralism was arguably brought out more clearly by a philosopher: Ernst Cassirer. Hence we can use Cassirer's writings to highlight it further. Particularly relevant in this context are his early article, "Kant und die moderne Mathematik" (1907), and his first systematic book, *Substanzbegriff und Funktionsbegriff* (1910).

Cassirer was, in fact, sensitive to the structuralist as well as the logicist side of Dedekind's approach. Thus, with respect to his 1872 essay he writes:

> We thus see [in Dedekind's writings] that, to get to the concept of irrational number, we do not need to consider the intuitive geometric relationships of magnitudes, but can reach this goal entirely within the arithmetic realm. A number, considered as part of a certain ordered system, consists of nothing more than a position ([4], p. 49, my translation).

Note here the recognition of Dedekind's conception of a real number as "nothing more than a position", but also the rejection of the need to appeal to geometrically based magnitudes. Concerning the natural numbers, Cassirer writes similarly:

> [Dedekind showed that] in order to provide a foundation for the whole of arithmetic, it is sufficient to define the number series simple as the succession of elements related to each other by a certain order—hereby thinking of the individual numbers, not as 'pluralities of units', but as characterized merely by the 'position' they occupy within the whole series ([4], p. 46, my translation).

With the characterization of the natural numbers as "pluralities of units", Cassirer points to an older tradition related to the appeal to magnitudes, a tradition that goes back to Euclid. In his 1910 book, he describes Dedekind's general, more abstract alternative succinctly thus:

> What is at issue is this: that there is a system of ideal objects whose content is exhausted in their mutual relations. The 'essence' of numbers consists in nothing more than their positional value ([5], p. 39, my translation).

Cassirer does not use the term "structuralism" yet (nor does Dedekind). But the basic idea is clear enough: Along Dedekindian lines, mathematical objects have come to be understood as mere "positions" in relational systems or structures.

The germ for this idea can again be found in Gauss, who wrote that what is crucial in mathematics are not objects but relations, relations of relations, etc. (cf. [13]). But with Dedekind's systematic re-thinking of the natural and real numbers the idea has been elaborated much more clearly. The result is, as Cassirer highlights too, that mathematics has been transformed from the traditional "science of number and quantity" into the much more general study of "relationally" or "functionally" characterized systems. In fact, Cassirer puts this transformation into an even broader context, namely the shift from a primarily "substance-based" perspective, exemplified by Aristotelian logic and pre-modern science, to "function-based" thinking, exemplified by Galilei, Newton, Einstein, and finding a paradigmatic expression in Dedekind's approach to pure mathematics.

6 Dedekindian Themes in Current Philosophy of Mathematics

Dedekind's technical results have been celebrated as important for the foundations of mathematics for quite a while. In contrast, his philosophical remarks—brief, cryptic, and sprinkled though his writings—were long ignored or received very critically, with the exception of Cassirer's works (cf. [24]). This situation has changed again during the last few decades, at least to some degree. Let us start once more with Dedekind's logicism.

Often identified with Frege's and Russell's versions of it, logicism fell out of favor in the mid-twentieth century, primarily because of Russell's antinomy and related problems. Since the 1980s there has been a revival, however, initiated by Crispin Wright, Bob Hale, and other "neo-logicists". Their approach is based on "abstraction principles" similar to, but more restricted and less problematic than, Frege's original "Basic Law V" (for classes). The resulting neo-logicism has been developed in considerable technical detail recently (cf. [12]). However, it has not found much echo in mainstream mathematics yet, since it is far removed from current mathematical practice.

On the other hand, there are interesting connections to Dedekind's remarks about abstraction and to the mathematical practice of his time. In fact, it might be possible to develop a Dedekindian form of neo-logicism, parallel to the usual neo-Fregean form, in similar technical detail. This is a live philosophical project (cf. [19], [26], for historical background [20]).

While such a project attempts to reconstruct Dedekind's form of logicism in novel, technically sophisticated ways, there is also a debate in the current literature about whether the historical Dedekind should be understood as a "logicist" in the first place. People on one side of that debate emphasize his relevant programmatic remarks, as I did above, together with offering interpretations of their philosophical underpinnings ([15], [18], [23]). People on the other side stress differences between Dedekind's approach and those by Frege and Russell, while suggesting alternative philosophical readings of him ([2]). For the second group, a main argument is that Dedekind was not concerned about static foundations, as Frege and Russell were, but about developing mathematics further. But as my remarks about Dedekind's logicism above indicate, the two sides might not be as far apart as it seems. In any case, more remains to be said about this issue.

In addition to the rise of neo-logicism, there has also been a revival of structuralist ideas in the philosophy of mathematics, in this case starting in the 1960s, as initiated by Paul Benacerraf and others, but picking up steam again in the 1980s. "Structuralism" is here understood as the view that mathematics is the study of "abstract structures", while we can leave aside the "intrinsic nature" of objects exemplifying them. In works by Geoffrey Hellman, Charles Parsons, Stewart Shapiro, etc., this slogan has been elaborated in a variety of ways, leading to both "eliminative" and "non-eliminative" versions of structuralism (cf. [21]). In the corresponding literature, Dedekind is usually re-appropriated positively. In earlier work of mine ([22]), I argue that Dedekind's position is interestingly different from all the options introduced more recently. In particular, I attribute a distinctive form of "non-eliminative structuralism" to him, one that takes his remarks about "creation" seriously while interpreting them in a relatively deflationary way.

This interpretation of Dedekind's structuralism has been challenged in turn (cf. [30]). Or more precisely, the new counter-suggestion is that Dedekind changed his relevant views over time, with the effect that he arrived at a form of "eliminative structuralism" in the 1880s. The latter amounts to the view according to which any system satisfying the Dedekind-Peano axioms can be regarded as "the natural numbers", while we do not need to introduce a separate, privileged such system, as Dedekind's remarks about "creation" seem to indicate; similarly for the real numbers. At the core of that position is a form of "relative" or "pragmatic" identification of isomorphic systems that is implicit in much of modern mathematical practice (cf. [27]). Hence it is attractive to attribute it to Dedekind already. But questions remain about whether the interpretation can really do justice to Dedekind's original remarks, and a "non-eliminative" position has its own attractions. Here too, we have not heard the last word about Dedekind yet.

7 Conclusion

I started with some reasons for skepticism about viewing Dedekind as a philosopher of mathematics. The rest of the article provided a sustained case against such skepticism. Not only are Dedekind's many technical contributions to the foundations of mathematics undoubtedly important; and not only are the ways in which he systematically re-thought various parts of mathematics significant. There are also several terse but philosophically pregnant remarks in Dedekind's writings that, if interpreted in context, reveal themselves as the tip of a philosophical iceberg. This was illustrated with respect to two main themes (a list not meant to be exhaustive): Dedekind's logicism and his structuralism.

As influenced by his teachers, Gauss, Dirichlet, and Riemann, Dedekind pursued the project of systematically re-thinking the "pure mathematics" that had arisen in the nineteenth century. This re-thinking took on a logicist and structuralist form, as noted early on by Ernst Cassirer. It also makes his work relevant for several debates in current philosophy of mathematics. The discussion of the latter had to be rather brief and sketchy (with references to further literature). Altogether, the conclusion is this: With his subtle, but often largely implicit, responses to philosophical issues arising out of mathematical developments Dedekind remains an important philosopher of mathematics today.

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References

[1] Awodey S., Reck E.: Categoricity and Completeness, Part 1: Nineteenth-Century Axiomatics to Twentieth-Century Metalogic. *History and Philosophy of Logic* 23: 1–30 (2002)

[2] Benis-Sinaceur H., Panza M., Sandu G.: *Functions and Generality of Logic: Reflections on Dedekind's and Frege's Logicisms*. Berlin, Springer (2015)

[3] Bolzano B.: *Paradoxien des Unendlichen*. Leipzig, Reclam (1851)

[4] Cassirer E.: Kant und die moderne Mathematik. *Kant-Studien* 12: 1–49 (1907)

[5] Cassirer E.: *Substanzbegriff und Funktionsbegriff*. Berlin, Bruno Cassirer (1910)

[6] Dedekind R.: Über die Einführung neuer Funktionen in der Mathematik: Habilitationsvortrag (1854). In: *Gesammelte Werke*, vol. 3, 428–438.

[7] Dedekind R.: *Stetigkeit und irrationale Zahlen*. Braunschweig, Vieweg (1872). Reprinted in *Gesammelte Werke*, vol. 3, 315–334. English trans. Continuity and Irrational Numbers. In [10], 1-27.

[8] Dedekind R.: *Was sind und was sollen die Zahlen*. Braunschweig, Vieweg (1888). Reprinted in *Gesammelte Werke*, vol. 3, 335–391. English trans. The Nature and Meaning of Numbers. In [10], 44–113.

[9] Dedekind R.: *Gesammelte Werke*, vol. 1–3. Braunschweig, Vieweg (1930–32)

[10] Dedekind R.: Essays on the Theory of Numbers. New York, Dover (1963)

[11] Dugac P.: *Richard Dedekind et les Fondements des Mathématiques*. Paris, Vrin (1976)

[12] Ebert P., Rossberg M.: *Abstractionism: Essays in Philosophy of Mathematics*. Oxford, Oxford University Press (2017)

[13] Ferreirós J.: The Birth of Pure Mathematics, as Arithmetic in Gauss. In Goldstein C. and Schappacher N. and Schwermer J. (eds.): *The Shaping of Arithmetic after C.F. Gauss's* Disquisitiones Arithmeticae. Berlin, Springer (2007), 235–268.

[14] Ferreirós J.: The Early Development of Set Theory. *Stanford Encyclopedia of Philosophy*. (2016; previous versions online since 2007)

[15] Ferreirós J.: On Dedekind's Logicism. In Arana A., Alvarez C. (eds.): *Analytic Philosophy and the Foundations of Mathematics*. London, Palgrave (2017)

[16] Frege G.: *Die Grundlagen der Arithmetik*. Breslau, Koeber (1884)

[17] Hallett M.: Foundations of Mathematics. In Baldwin T. (ed.): *Cambridge History of Philosophy, 1870–1945*. Cambridge University Press (2003), 128–154.

[18] Klev A.: Dedekind's Logicism. *Philosophia Mathematica* (2015)

[19] Linnebo O., Pettigrew R.: Two Types of Abstraction for Structuralism. *Philosophical Quarterly* 64: 267–283 (2014)

[20] Mancosu P.: Abstraction and Infinity. Oxford University Press (2017)

[21] Parsons C.: The Structuralist Conception of Mathematical Objects. *Synthese* 84: 303–346 (1990)

[22] Reck E.: Dedekind's Structuralism: An Interpretation and Partial Defense. *Synthese* 137: 369–419 (2003)

[23] Reck E.: Frege, Dedekind, and the Origins of Logicism. *History and Philosophy of Logic* 34: 242–265 (2013)

[24] Reck E.: Frege or Dedekind? Towards a Reevaluation of their Legacies. In Reck E. (ed.): *The Historical Turn in Analytic Philosophy*. London, Palgrave (2013), 139–170.

[25] Reck E.: Dedekind's Contributions to the Foundations of Mathematics. *Stanford Encyclopedia of Philosophy*. (2016; previous versions online since 2008)

[26] Reck E.: On Reconstructing Dedekind Abstraction Logically. In Reck E. (ed.): *Logic, Philosophy of Mathematics, and their History: Essays in Honor of W.W. Tait*. London, College Publications (2017)

[27] Reck E., Price M.: Structures and Structuralism in Contemporary Philosophy of Mathematics. *Synthese* 125: 341–383 (2000)

[28] Sieg W.: Searching for Proofs (and Uncovering Capacities of the Mathematical Mind). In Feferman S. and Sieg W. (eds.): *Proofs, Categories, and Computations: Essays in Honor of Grigory Mints*, College Publications, 189–2015 (2010)

[29] Sieg W., Schlimm D.: Dedekind's Analysis of Number: Systems and Axioms. *Synthese* 147: 121–170 (2005)

[30] Sieg W., Morris R.: Dedekind's Structuralism: Creating Concepts and Proving Theorems. In Reck E. (ed.): *Logic, Philosophy of Mathematics, and their History: Essays in Honor of W.W. Tait*. London, College Publications (2017)

[31] Stein H.: *Logos*, logic, *logistiké*: Some Philosophical Remarks on Nineteenth-Century Transformations of Mathematics. In Aspray W., Kitcher P. (eds.): *History and Philosophy of Modern Mathematics*. Minneapolis, MN, University of Minnesota Press (1988)